Maximum Likelihood Convolutional Decoding (MCD) Performance Due to System Losses

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A model for predicting the computational performance of a maximum likelihood convolutional decoder (MCD) operating in a noisy carrier reference environment is described. This model is used to develop a subroutine that will be utilized by the Telemetry Analysis Program (TAP) to compute the MCD bit error rate. When this computational model is averaged over noisy reference phase errors using a high-rate interpolation scheme, the results are found to agree quite favorably with experimental measurements.

I. Introduction

The maximum likelihood or Viterbi decoding algorithm was discovered and analyzed by Viterbi (Ref. 1) in 1967. Maximum likelihood convolutional decoding (MCD) using the Viterbi decoding algorithm is presently being implemented in the Deep Space Network (DSN).

In order to develop specifications and criteria for executing system performance tests, it was necessary to develop a program to provide a prediction of the MCD's performance. This program is to be integrated into the Telemetry Analysis Program (TAP).

Using curve fitting techniques on data produced by the baseband characteristic performance curve (Fig. 1) the model predicts MCD performance under noisy carrier reference conditions.

II. Discussion of MCD Performance Prediction Model (One-Way Radio Loss)

In developing the MCD performance prediction model, we begin with the baseband performance characteristic curve for maximum likelihood convolutional decoding of a $K = 7$, rate 1/2 convolutional code with $Q = 3$ (Ref. 2). In general, the characteristic curves here mentioned represent the best baseband estimate of the MJS'77 code performance under ideal (i.e., laboratory) conditions. As such, the characteristics represent an upper bound on telemetry system performance. However, owing to the fact that there is no exact analytical expression for the performance characteristic, we have to use measured baseband performance data and, noting the fact that these data, as well as the simulation results of the MCD (Ref. 3, Section 4), define a relationship between P_e and

E_b/N_0, numerically approximate an expression for the MCD performance characteristics. Written formally,

$$P_e = f\left(\frac{E_b}{N_0}\right) \tag{1}$$

for a given code, receiver quantization, and Viterbi decoder, where P_e denotes the probability of bit error and E_b/N_0 is the ratio of signal energy per bit to noise spectral density.

An inaccurate carrier phase reference at the demodulator will degrade system performance (see Fig. 2). In particular, a constant error ϕ in the demodulator phase will cause the signal component of the matched filter output to be suppressed by the factor cos ϕ (Ref. 3, Section 5).

$$r_i = I\sqrt{\frac{2E_s}{N_0}}\cos\phi + n_i \tag{2}$$

Since the carrier phase is being tracked in the presence of noise, the phase error ϕ will vary with time. When the data rate is large compared to the carrier loop bandwidth, the carrier phase error ϕ does not vary significantly during perhaps 20–30 information bit times. Therefore, the phase error is assumed to be constant over the length of almost any decoder error. This being the case, the bit error probability for a constant phase error ϕ can be written as

$$P_e(\phi) = f\left(\frac{E_b}{N_0}\cos^2\phi\right) \tag{3}$$

from (1) and (2), making use of the fact that the received signal energy is degraded by $\cos^2\phi$.

Let $\phi(t)$ be the phase error in the receiver phase locked loop (PLL). The phase error $\phi(t)$ is an ergodic random process whose probability density function (for a second order PLL) may be written as

$$P(\phi) = \frac{\exp(\alpha\cos\phi)}{2\pi I_0(\alpha)}, \qquad \alpha >> 1 \tag{4}$$

where α is the signal-to-noise ratio in the carrier phase tracking loop, and $I_0(\ )$ is the zeroth order modified Bessel function. See Ref. 4, pages 90 and 198, for derivation of $p(\phi)$. In Ref. 4, it is shown that for large α

$$I_0(\alpha) \sim \frac{\exp(\alpha)}{(2\pi\alpha)^{1/2}} \tag{5}$$

Therefore, we can write for the probability density function (PDF) of ϕ:

$$P(\phi) \sim \left(\frac{\alpha}{2\pi}\right)^{1/2}\exp(\alpha\cos\phi - \alpha) \tag{6}$$

If we let $P(E)$ denote the resulting probability of bit error when considering the performance of the MCD when the carrier phase is being tracked in the presence of noise (imperfect carrier phase reference), where ϕ is a random variable with PDF $p(\phi)$, Eq. (6),

$$P(E) = \int_{-\pi}^{\pi} P(\phi)\,\mathrm{P}_e(\phi)\,d\phi \tag{7}$$

Since $p(\phi)$ is known as a function of loop SNR, the joint density $P\ (E,\ \phi)$ is also known as a function of loop SNR and the dependence on ϕ can be integrated out, hence Eq. (7). However, since $P_e(\phi)$ does not have an exact analytical expression, we must use the Viterbi decoder measured performance data and obtain a model representing the functional relationship expressed in Eq. (3).

Recognizing the fact that the bit error rate curve (Fig. 1) is a semi-log plot; we can use the relationship used in Ref. 5,

$$Y = A\exp(BX) \tag{8}$$

to model $P_e(\phi)$.

The model is

$$P_e(\phi) = A\exp\ (BX) \tag{9}$$

where $X = E_b/N_0$ and A and B are constants (Ref. 6). The constants A and B were determined using the TYMSHARE program CURFIT (see Appendix for a discussion of the curve fitting).

$$A = 85.7501$$

$$B = -5.7230$$

Since E_b/N_0 is degraded by the factor $\cos^2\phi$ for imperfectly coherent performance,

$$P_e(\phi) = A\exp\left(B\left(\frac{E_b}{N_0}\right)\cos^2\phi\right) \tag{10}$$

Substituting Eqs. (6) and (10) into Eq. (7), we get

$$P(E) = \int_{-\pi}^{\pi}\left(\frac{\alpha}{2\pi}\right)^{1/2}\exp(\alpha\cos\alpha - \alpha)A\exp\left(B\left(\frac{E_b}{N_0}\right)\cos^2\phi\right)d\phi \tag{11}$$

Now, by performing the algebra on Eq. (11) we get the following:

$$P(E) = \left(\frac{2\alpha}{\pi}\right)^{1/2} A \int_0^{\pi} \exp\left[B\left(\frac{E_b}{N_0}\right)\cos^2\phi - \alpha(1 - \cos\phi)\right] d\phi \tag{12}$$

Equation (12) has been integrated numerically using a modified Romberg quadrature subroutine. The tabulated results are found in Table 1 with the plot of the MCD performance predicting curves with the carrier phase tracking loop SNR α as a parameter (Fig. 2). Figure 3 shows the error rate performance requirement.

III. Comparison of Predicted and Experimental MCD Performance

The real test of a model is its ability to predict performance under real system operation conditions. In order to make a valid comparison, MCD performance data taken from NASA's Deep Space Network tracking stations must be compared with the model. Due to the fact that, at present, stations throughout the Network have barely begun to install and test the MCD, only a small amount of data are available with which to compare the model. In this paper, the model will be compared to data received from the Compatibility Test Area (CTA 21) and DSS 62 (Madrid, Spain).

The data from DSS 62 show in Fig. 4 (see Fig. 5 of Ref. 7, page 27) the behavior of the bit errors as a function of modulation indices. From this graph one can establish the optimum point of performance by choosing (for each P_t/N_0) that carrier suppression yielding the lowest error rate. As can be seen from Fig. 4, there exists a minimum degradation point at approximately 70 ± 1 deg modulation index.

In the $\Delta E_b/N_0$ column of Table 1 it is shown that the MCD performance prediction model compares very well to the actual MCD performance at DSS 62 and DSS 63 (Spain) for modulation indices of 69–71 deg. At a mod index of 70 ± 1 deg the model predicts to within 0.2 dB of the actual MCD performance on the average.

Table 2 shows the range of data taken at DSS 62/63. Column 1 of this table shows that the carrier phase tracking loop SNR ranges from 13.0 to 22.1 dB. By taking an average of the $\Delta E_b/N_0$ column, it is shown that the MCD prediction model predicts to within 0.25 dB of the actual MCD performance over this entire range of data.

Figures 5 and 6 show how well the MCD model compares with the tests performed at CTA 21. A more detailed comparison will be performed when more data are available from CTA 21.

IV. Telemetry Analysis Program (TAP) and the MCD Subroutine

The Network Operations System Support Group maintains a telemetry system analysis program that must be updated to include the MCD performance prediction subroutine. The performance prediction model will be incorporated in the Telemetry Analysis Program (TAP) as a subroutine. The conversational TAP has the capability of analyzing both block-coded and uncoded data for the Viking mission and uncoded data for the Pioneer and Helios missions. Block III as well as Block IV configurations are available. For a given receiver subcarrier demodulator assembly (SDA) and symbol synchronizer assembly (SSA) configuration (Blk III or Blk IV bandwidth settings), bit rate, modulation index, and ST_s/N_0, the program outputs telemetry performances in the form of receiver, SDA and SSA degradation. Following the adding of the MCD subroutine, the TAP will then provide a predicted performance degradation of signal due to carrier phase jitter and tracking loop SNR. It will also provide overall telemetry degradation as related to the addition of the MCD.

The items in Fig. 7 labeled "new" are essentially the changes necessary to implement the Telemetry Analysis Program processing steps to include the MCD update (see Fig. 8).

V. Conclusions

We have seen that the model of the maximum likelihood convolutional decoder (MCD) works quite well in predicting the performance of the on-station MCD. Table 1 shows that at an optimum mod index of 71 deg the $\Delta E_b/N_0$ (dB) between the performance prediction model and the actual data taken from DSS 62/63 (Spain) is approximately 0.17 dB average. Further comparisons will be made when we receive data from the other network stations.

References

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Table 1. Comparison of the MCD performance data from DSS 62/63 and the MCD performance prediction model for optimum mod index selected from Fig. 4

Mod index	Carrier phase tracking loop SNR, dB	Bit error rate	Energy per bit to noise spectral density ratio (Spain)	Energy per bit to noise spectral density ratio (prediction model)	$\Delta E_b/N_0$, dB
70	14.62	0.204×10^{-4}	4.43	4.58	0.15
	14.72	0.250×10^{-4}	4.25	4.55	0.30
	15.18	0.448×10^{-5}	4.66	5.00	0.34
71	14.24	0.536×10^{-4}	4.53	4.45	−0.08
	14.33	0.388×10^{-4}	4.38	4.51	0.13
	14.37	0.340×10^{-4}	4.23	4.50	0.27
	14.77	0.547×10^{-5}	4.86	5.07	0.21
69	15.00	0.119×10^{-4}	4.28	4.70	0.42
	15.09	0.800×10^{-5}	4.39	4.81	0.42
	15.14	0.345×10^{-4}	3.985	4.30	0.30
	15.57	0.355×10^{-5}	4.71	5.00	0.29

Table 2. Comparison of the MCD performance data from DSS 62/63 and the MCD performance prediction model

Carrier phase tracking loop SNR, dB	Bit error rate	Energy per bit to noise spectral density ratio (Spain)	Energy per bit to noise spectral density ratio (model)	$\Delta E_b/N_0$, dB
13.0	0.686×10^{-4}	4.40	5.1	0.7
13.5	0.202×10^{-4}	4.86	5.31	0.45
14.0	0.471×10^{-4}	4.39	4.55	0.16
14.4	0.633×10^{-5}	4.87	5.1	0.23
15.00	0.119×10^{-4}	4.28	4.7	0.42
15.57	0.355×10^{-5}	4.71	5.00	0.29
17.08	0.655×10^{-5}	4.51	4.6	0.09
19.79	0.413×10^{-3}	3.31	3.31	0.0
20.50	0.792×10^{-4}	3.83	3.85	0.02
22.10	0.837×10^{-5}	4.61	4.48	−0.13

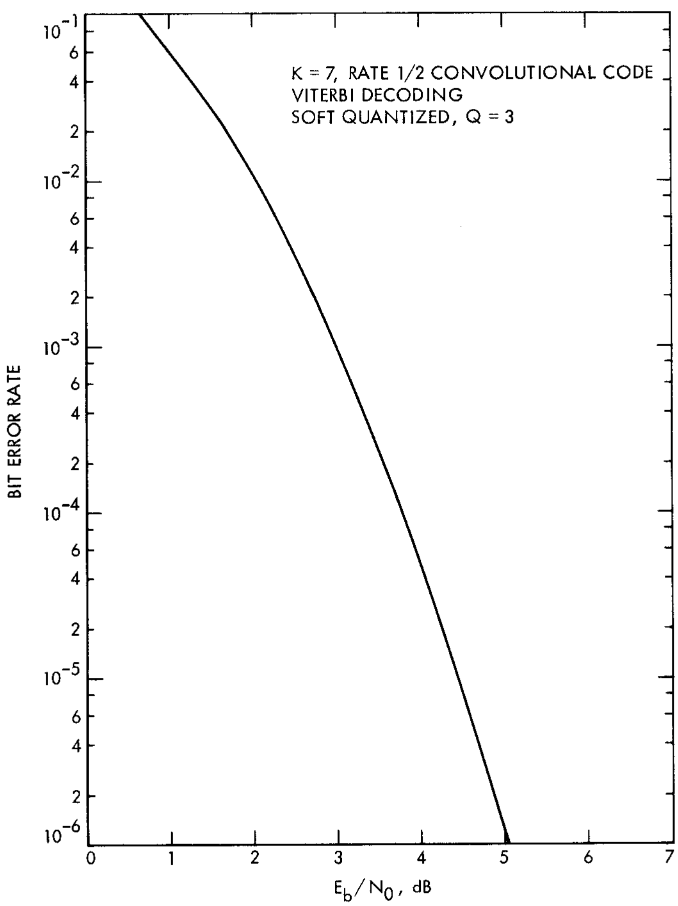

Fig. 1. Baseband performance characteristic

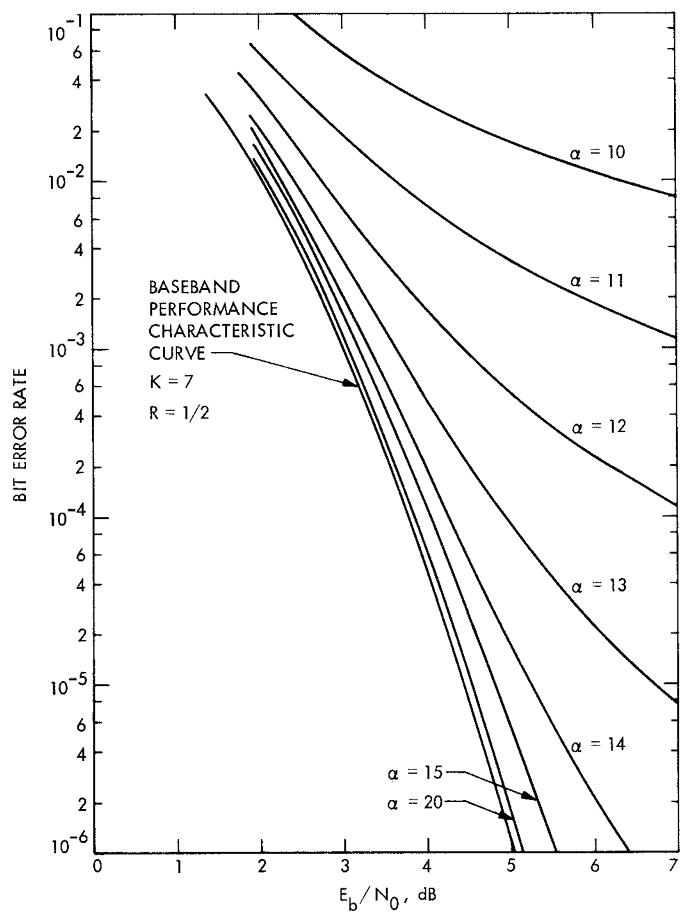

Fig. 2. MCD model output

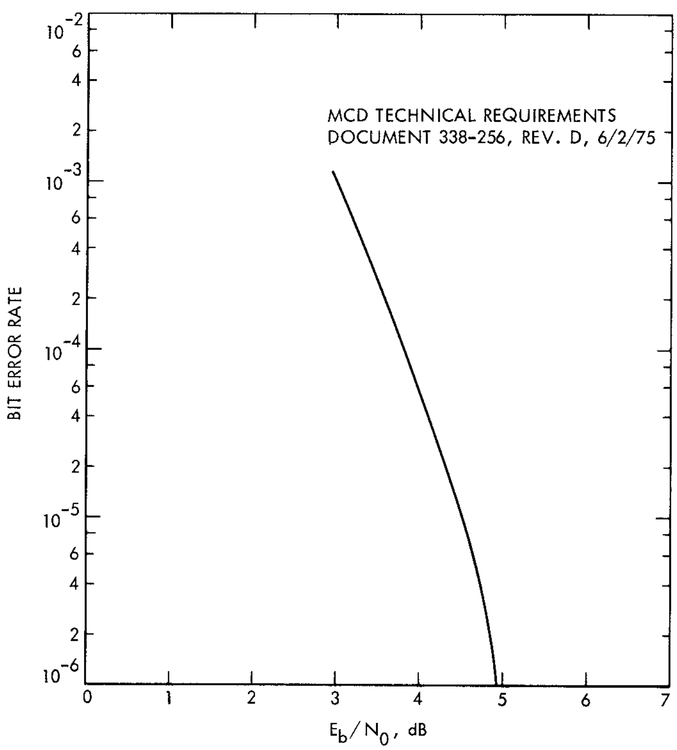

Fig. 3. Error rate performance requirement

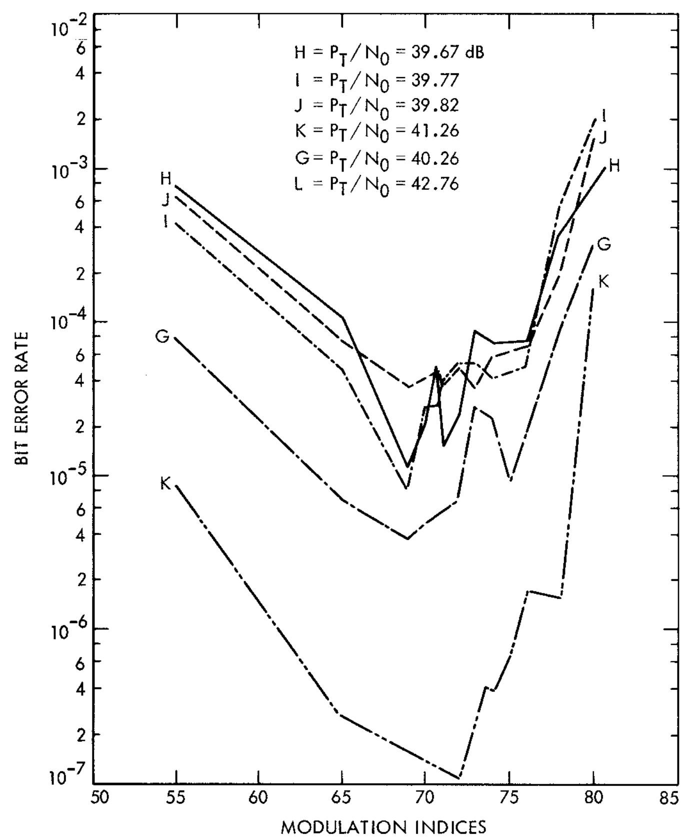

Fig. 4. BER vs modulation indices

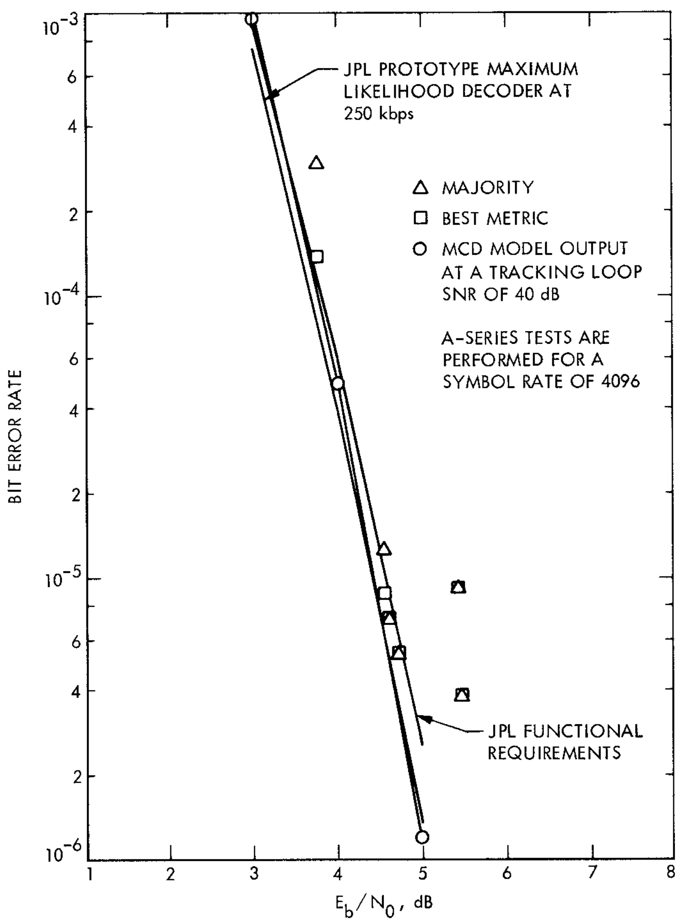

Fig. 5. E_b/N_0 **vs bit error probability, A-series test**

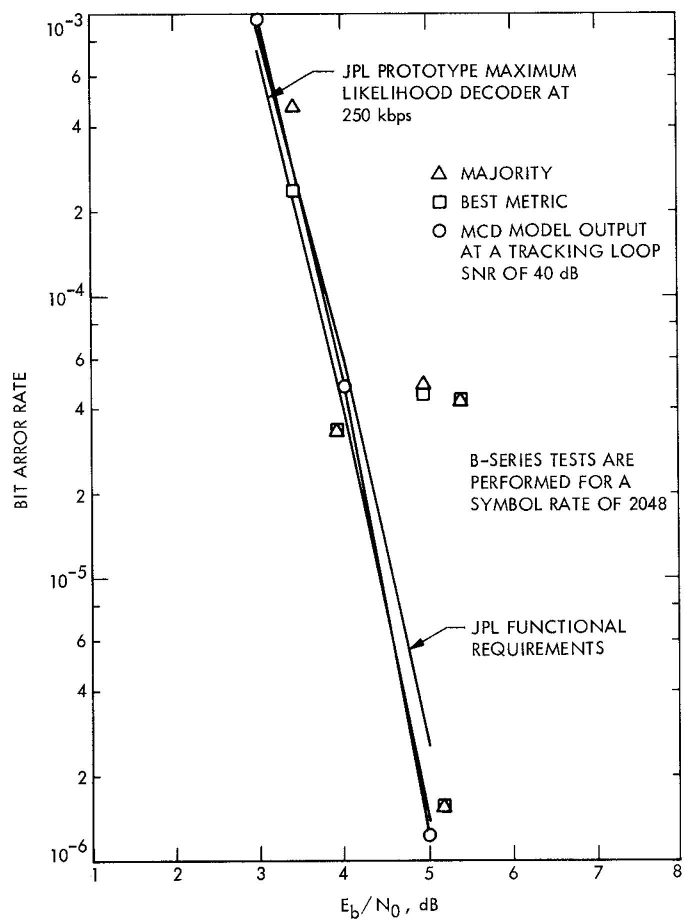

Fig. 6. E_b/N_0 **vs bit error probability, B-series test**

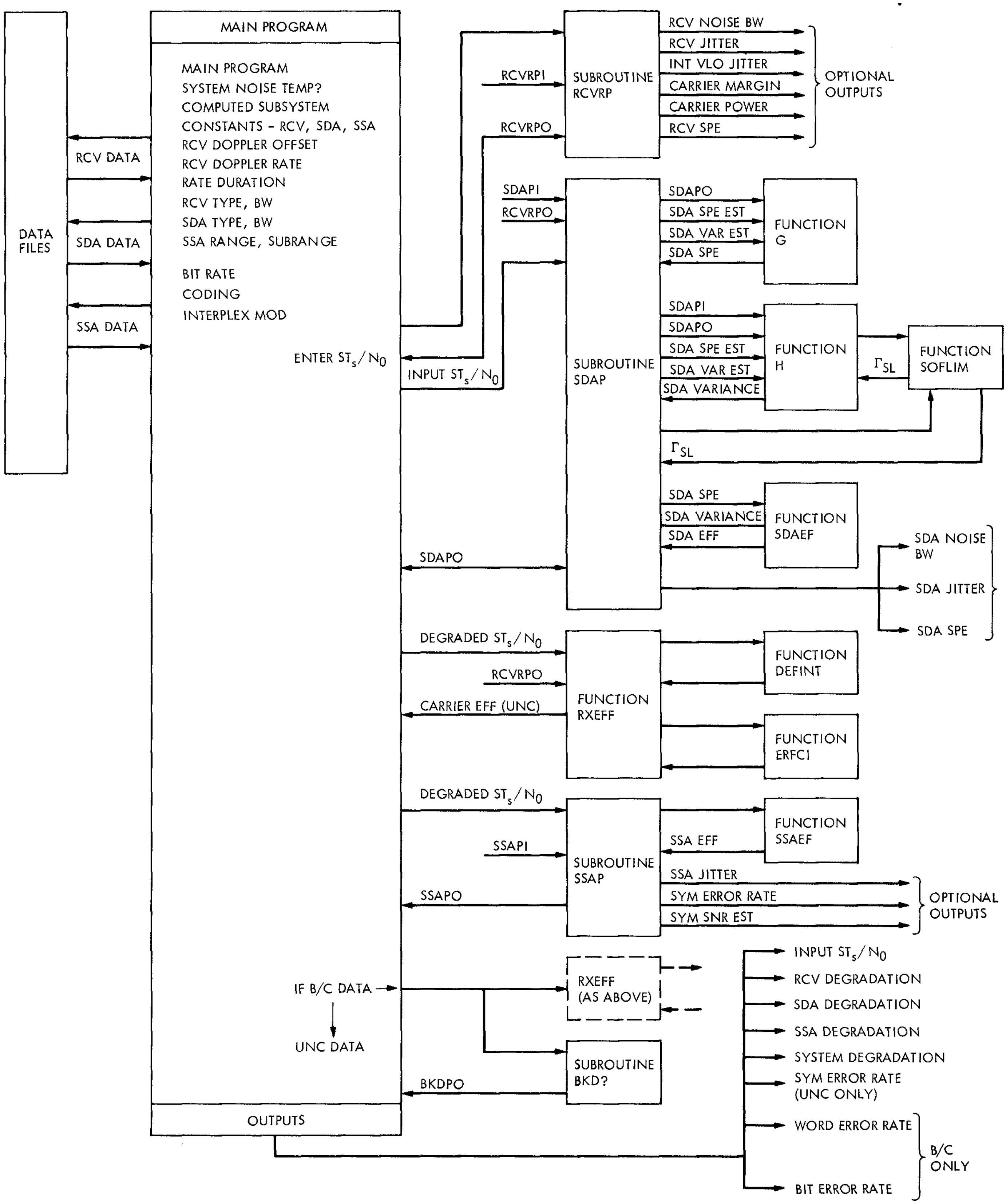

Fig. 7. Telemetry Analysis Program processing steps

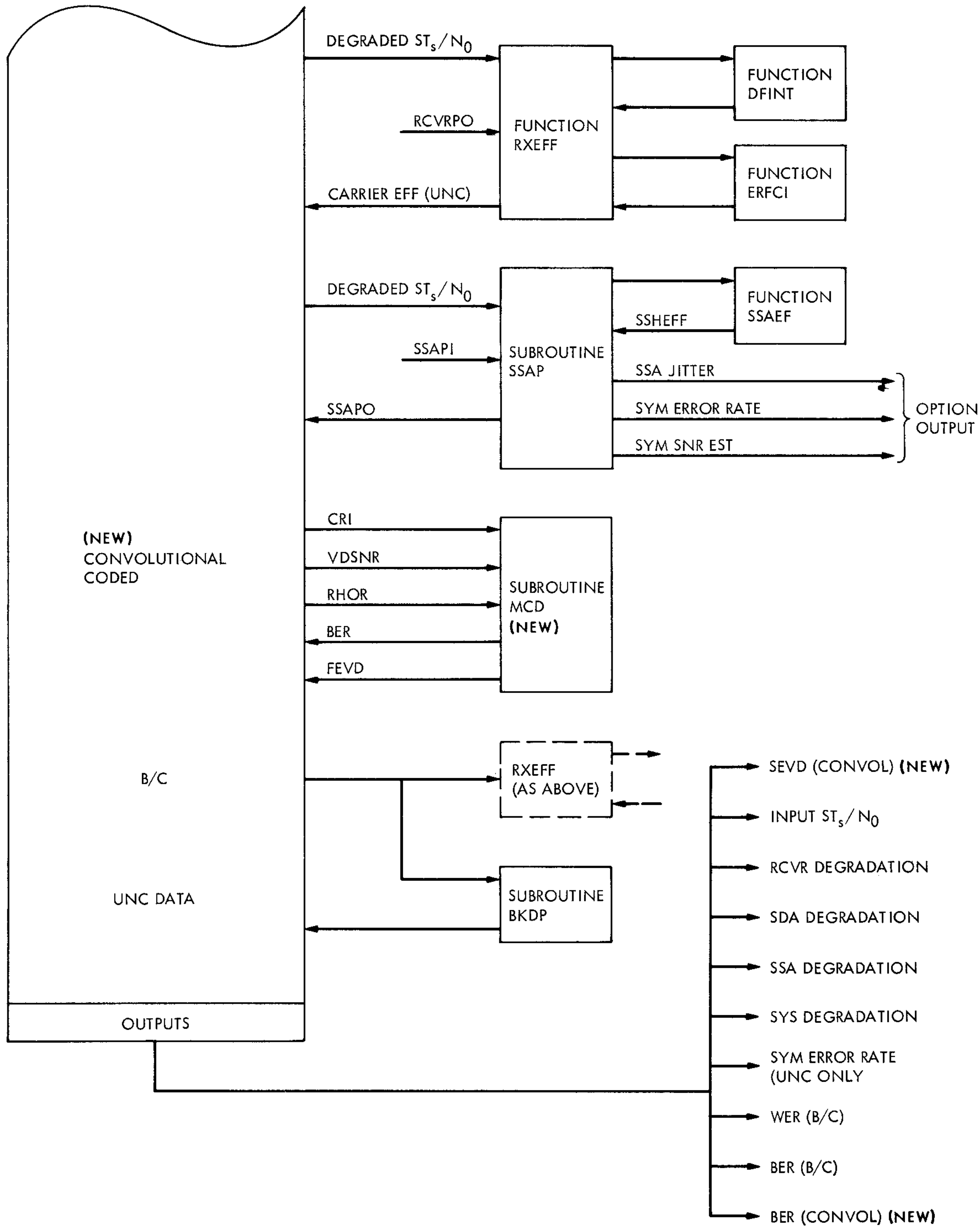

Fig. 8. Telemetry Analysis Program processing steps, MCD update

Appendix

CURFIT is a linear regression program for data with two variables, X and Y. It accepts as many as 200 observations on two variables. The program CURFIT determines which of the following curves can best approximate a set of input points described by X and Y coordinates.

(a) $Y = a + bx$

(b) $Y = ae^{bx}$

(c) $Y = ax^b$

(d) $Y = a + \dfrac{b}{x}$

(e) $Y = \dfrac{1}{a + bx}$

(f) $Y = \dfrac{x}{ax + b}$

Equation (b) best approximated the input points described by E_b/N_0 vs bit error rates (BER) taken from the baseband characteristic curve of Fig. 1.

Recognizing the fact that for convolutional coding with phase-coherent demodulation and Viterbi decoding, exact analytical expressions for bit error rate P_e vs E_b/N_0 are not attainable, we would like to establish the suitability of the empirical function:

$$Y = ae^{bx} \tag{A-1}$$

that has been determined by CURFIT as yielding the best approximation for BER vs E_b/N_0.

One method of determining the suitability of the empirical function is that of finite differences. This method relies principally on the hypothesis that, for a given function, differences of the function $Y = ae^{bx}$ are constant for successive equal increments of some known function of X.

For

$$P(E) = Ae^{BX} \tag{A-2}$$

where A and B are constants and $X = E_b/N_0$, suppose that ΔX is a small constant incremental change in X and $\Delta P(E)$ is the corresponding change in $P(E)$. Then, corresponding to an increased change from X to $(X + \Delta X)$,

$$P(E) + \Delta P(E) = AC^{B(X+\Delta X)} \tag{A-3}$$

must also hold true if this is a suitable empirical function.

If we subtract from $[P(E) + \Delta P(E)]$ the value $P(E)$ we get the following:

$$[P(E) + \Delta P(E)] - P(E) = Ae^{B(X+\Delta X)} - Ae^{BX}$$

$$\Delta P(E) = Ae^{BX}e^{BX} - Ae^{BX}$$

$$\Delta P(E) = Ae^{BX}(e^{B\Delta X} - 1) \tag{A-4}$$

Taking the natural logarithm of both sides

$$\ln \Delta P(E) = \ln A + \ln e^{BX} + \ln (e^{B\Delta X} - 1) \tag{A-5}$$

But ΔX is constant by supposition, and therefore in $(e^{B\Delta X} - 1)$ is a constant, independent of X (but not ΔX); hence, we can rewrite $\ln \Delta P(E)$ in the following way;

$$\ln \Delta P(E) = A^1 + BX \tag{A-6}$$

where $\ln e^{BX} = BX$, and $A^1 = \ln A + \ln (e^{B\Delta X} - 1)$.

Let us take an incremental change in $\ln \Delta P(E)$ denoted as follows:

$$\ln \Delta P(E) + \Delta[\ln \Delta P(E)] \tag{A-7}$$

and subtract this new identity (Eq. A-6); we get

$$(\ln \Delta P(E) + \Delta[\ln \Delta P(E)] - [\ln \Delta P(E)] = [A^1 + B(X + \Delta X)] - [(A^1 + BX)] \tag{A-8}$$

Hence

$$\Delta[\ln \Delta P(E)] = B\Delta X \tag{A-9}$$

Thus, for this type of relationship, the differences of the logarithm of the first differences will be constant. As an added advantage, Eq. (A-9) can be used in the determination of the coefficient B by averaging or using a mean-square average of the factors $\Delta[\ln \Delta P(E)]$.